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Efficient Finite Difference Design Sensitivities

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The problem considered in this study is to evaluate efficiently displacement derivatives using global finite differences. Given the displacements for an initial design, the displacements for various modified designs are evaluated by the recently developed combined approximations method. Calculations of finite difference sensitivity coefficients are demonstrated for static problems and eigenproblems. The presented solution procedure is easy to implement, efficient, and can be used to calculate derivatives for various designs where the exact displacements are not known. Some numerical examples show that the accuracy of the results is similar to the accuracy obtained by finite difference calculations based on exact analysis.

Nomenclature

$\boldsymbol{B}$	=	matrix defined by Eq. (12)
E	=	true percentage error
EI	=	bending stiffness
$\boldsymbol{K}$	=	stiffness matrix
$\boldsymbol{K}_R$	=	reduced stiffness matrix
L	=	length of element
$\boldsymbol{M}$	=	mass matrix
$\boldsymbol{M}_R$	=	reduced mass matrix
n	=	number of elements
$\boldsymbol{R}$	=	load vector
$\boldsymbol{R}_R$	=	reduced load vector
$\boldsymbol{r}$	=	displacement vector, basis vector, mode shape
$\boldsymbol{r}_B$	=	matrix of basis vectors
s	=	number of basis vectors
$\boldsymbol{U}$	=	upper triangular matrix
X	=	design variable
$\boldsymbol{y}$	=	vector of coefficients
α	=	step size variable
β	=	angle between vectors
$\Delta\boldsymbol{K}$	=	change in the stiffness matrix
$\Delta\boldsymbol{M}$	=	change in the mass matrix
$\Delta\boldsymbol{R}$	=	change in the load vector
ΔX	=	change in a design variable X
δX	=	perturbation in a design variable X
ε	=	error vector
λ	=	eigenvalue

I. Introduction

DESIGN sensitivity analysis of structures deals with the calculation of changes in the response resulting from changes in the parameters describing the structure. The derivatives of the response vector with respect to the system parameters, called the sensitivity coefficients, are used in the solution of various problems. In design optimization, the sensitivity coefficients are used to select a search direction. Their calculation often involves much computational effort, particularly in the optimization of large structural systems. Moreover, the calculation of derivatives for a given design involves structural analysis of the design. As a result, there has been much interest in efficient procedures for calculating the sensitivity coefficients.[1]

Design sensitivities are often used in generating approximations for the response of a modified system, including approximate reanalysis models and explicit approximations of the constraint functions in terms of the structural parameters (e.g., first-order Taylor-series approximations). In addition, the sensitivities are required for assessing the effects of uncertainties in the structural properties (e.g., material or geometric properties) on the system response.

In general, the following factors are considered in choosing a suitable sensitivity analysis method for a specific application: 1) the accuracy of the calculations, 2) the computational effort involved, and 3) the ease of implementation. The implementation effort is weighted against the performance of the algorithms as reflected in their computational efficiency and accuracy. The quality of the results and the ease-of-implementation are usually two conflicting factors. That is, better results are often achieved at the expense of more implementation effort.

The present study deals with design sensitivity analysis for discrete systems. The two general approaches used for calculating the sensitivity coefficients are as follows: 1) the direct approach, which is based on the implicit differentiation of the analysis equations that describe the system response with respect to the desired parameters, and the solution of the resulting sensitivity equations; and 2) the adjoint-variable approach, where an adjoint physical system is introduced whose solution permits the rapid evaluation of the desired sensitivity coefficients. The adjoint-variable approach is not considered in this study because it is not available as an option in global finite difference derivatives.

Methods of design sensitivity analysis for discretized systems can be divided into the following classes:

1) The first class is finite difference methods, which are easy to implement but might involve numerous repeated analyses and involve high computational cost, particularly in problems with many design or response variables. In addition, finite difference approximations might have accuracy problems. The efficiency can be improved by using fast reanalysis techniques.

2) The second is analytical methods, which provide exact solutions but might not be easy to implement in some problems such as shape optimization.

3) The third class is semi-analytical methods, which are based on a compromise between finite difference methods and analytical methods. These methods use finite difference evaluation of

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the right-hand-side vector. They are easy to implement but might provide inaccurate results.[2,3] The accuracy problems can be eliminated using the exact semi-analytical formulation or the refined formulation.

4) The last class is computational or automated derivatives that rely on differentiation of the software itself.

Approximation concepts are often used to reduce the computational cost involved in repeated analysis of structures.[4] However, most approximations that are adequate for structural reanalysis are not sufficiently accurate for sensitivity analysis. In this study, approximate reanalysis is used to improve the efficiency of finite difference sensitivity analysis. Given the exact displacements for an initial design, the displacements for various modified designs are evaluated efficiently by the recently developed combined approximations method. Using the presented procedure, the displacement derivatives can be evaluated not only at the initial design but also at various designs where the exact displacements are not known.

Calculation of analytical derivatives using an approximate analysis models has been demonstrated previously.[5,6] It was found that accurate results can be achieved, but, as noted earlier, analytical derivatives might not be easy to implement. The presented solution procedure is easy to implement and can improve significantly the efficiency of the calculations. Some numerical examples show that the accuracy of the results is similar to the accuracy obtained by finite difference calculations based on exact analysis and on a refined semi-analytical method.

The outline of the paper is as follows. Global finite difference derivatives are briefly described in Sec. II. Sensitivity calculations for static problems and for eigenproblems are presented in Secs. III and IV, respectively. Efficiency considerations are discussed in Sec. V, some numerical examples are demonstrated in Sec. VI, and the conclusions are drawn in Sec. VII.

II. Global Finite Difference Derivatives

For simplicity the discussion is restricted to a single design variable. Consider the problem of calculating the derivatives $\partial \boldsymbol{r}_0/\partial X$ of the displacement vector $\boldsymbol{r}$ with respect to a design variable X at the point X_0. In the forward-difference method, the derivatives are approximated from the exact displacements at the original point X_0 and at the perturbed point $X_0 + \delta X$ by

$$\frac{\partial \boldsymbol{r}_0}{\partial X} = \frac{\boldsymbol{r}(X_0 + \delta X) - \boldsymbol{r}(X_0)}{\delta X} \tag{1}$$

where δX is a predetermined step size. The accuracy can be improved by adopting the central-difference approximation, where the derivatives are computed from the exact displacements at the two points $X_0 - \delta X$ and $X_0 + \delta X$ by

$$\frac{\partial \boldsymbol{r}_0}{\partial X} = \frac{\boldsymbol{r}(X_0 + \delta X) - \boldsymbol{r}(X_0 - \delta X)}{2\delta X} \tag{2}$$

Finite difference methods are the easiest to implement, and therefore they are attractive in many applications. When $\boldsymbol{r}(X_0)$ is known, application of Eq. (1) involves only one additional calculation of the displacements at $X_0 + \delta X$, whereas Eq. (2) requires calculation at the two points $X_0 - \delta X$ and $X_0 + \delta X$. For a problem with n design variables, finite difference derivative calculations require repetition of the analysis for $n + 1$ [Eq. (1)] or $2n + 1$ [Eq. (2)] different design points. This procedure is usually not efficient compared to, for example, analytical and semi-analytical methods.

As noted earlier, finite difference approximations might have accuracy problems. The following two sources of errors should be considered whenever these approximations are used:

1) The first is the truncation error, which is a result of neglecting terms in the Taylor-series expansion of the perturbed response.

2) The second is the condition error, which is the difference between the numerical evaluation of the function and its exact value. Examples for this type of error include round-off error in calculating $\partial \boldsymbol{r}_0/\partial X$ from the original and perturbed values of $\boldsymbol{r}$ and calculation of the response by approximate analysis. The latter can also be the result of a finite number of iterations being used within an iterative solution procedure.

These are two conflicting considerations. That is, a small step size δX will reduce the truncation error, but might increase the condition error. In some cases there might not be any step size that yields an acceptable error. Some considerations for choosing the forward-difference step size are discussed elsewhere.[7] In certain applications, truncation errors are not of major importance because it is often sufficient to find the average rate of change in the structural response and not necessarily the accurate local rate of change at a given point. Therefore, to eliminate round-off errors caused by approximations it is recommended to increase the step size.

It is well known that small response values are not calculated as accurately as large response values.[1] The same applies to derivatives. Thus, it would be difficult to evaluate accurately small response derivatives by finite difference or other approximations. Fortunately, it is usually not important to evaluate accurately small derivative values. The relative magnitude of the derivatives can be estimated from the ratio $(\partial \boldsymbol{r}/\boldsymbol{r})/(\partial X/X)$.

III. Sensitivity Calculations for Static Problems

A. Analytical Derivatives

Given a design X_0 and the corresponding symmetric, positive-definite stiffness matrix $\boldsymbol{K}_0$, the resulting displacements $\boldsymbol{r}_0$ are computed by the linear equilibrium equations

$$\boldsymbol{K}_0 \boldsymbol{r}_0 = \boldsymbol{R}_0 \tag{3}$$

where $\boldsymbol{R}_0$ is the load vector. In general, the stiffness matrix is first factorized by

$$\boldsymbol{K}_0 = \boldsymbol{U}_0^T \boldsymbol{U}_0 \tag{4}$$

where $\boldsymbol{U}_0$ is an upper triangular matrix. Once $\boldsymbol{U}_0$ is given, calculation of the displacements by Eq. (3) involves only forward and backward substitutions.

Differentiating Eq. (3) with respect to the design variable X and rearranging yields

$$\boldsymbol{K}_0 \frac{\partial \boldsymbol{r}_0}{\partial X} = \frac{\partial \boldsymbol{R}_0}{\partial X} - \frac{\partial \boldsymbol{K}_0}{\partial X} \boldsymbol{r}_0 \tag{5}$$

The right-hand side of this equation is often referred to as the pseudoload vector. Equations (3) and (5) have the same coefficient matrix $\boldsymbol{K}_0$. If the decomposed form of Eq. (4) is available from the analysis [Eq. (3)], then only forward and backward substitutions are needed to solve for $\partial \boldsymbol{r}_0/\partial X$ [Eq. (5)]. In many problems the load vector $\boldsymbol{R}$ is independent of the design variables. In such cases Eq. (5) is reduced to the form

$$\boldsymbol{K}_0 \frac{\partial \boldsymbol{r}_0}{\partial X} = -\frac{\partial \boldsymbol{K}_0}{\partial X} \boldsymbol{r}_0 \tag{6}$$

Analytical methods are widely used and often demonstrate good performance, in terms of both accuracy and efficiency. However, implementation of these methods is difficult in some problems such as shape optimization, where the derivatives in the right-hand-side vector of Eq. (5) are not easy to obtain. Employing finite difference methods can improve significantly the ease of implementation at the expense of less accurate results. How the finite difference approximations affect the computational efficiency strongly depends on the problem at hand.

B. Semi-Analytical Derivatives

Semi-analytical methods are intended to improve the ease of implementation of analytical methods. These methods are based on finite difference evaluation of the right-hand-side vector of Eq. (5). In the forward-difference method, the displacement derivatives $\partial \boldsymbol{r}_0/\partial X$ are computed from the exact values of $\boldsymbol{R}$ and $\boldsymbol{K}$ at the two points X_0 and $X_0 + \delta X$ by

$$\boldsymbol{K}_0 \frac{\partial \boldsymbol{r}_0}{\partial X} = \frac{\boldsymbol{R}(X_0 + \delta X) - \boldsymbol{R}(X_0)}{\delta X} - \frac{\boldsymbol{K}(X_0 + \delta X) - \boldsymbol{K}(X_0)}{\delta X} \boldsymbol{r}_0 \tag{7}$$

Using the central-difference method, the requested derivatives are computed from the exact values of $\boldsymbol{R}$ and $\boldsymbol{K}$ at the two points $X_0 - \delta X$ and $X_0 + \delta X$ by

$$\boldsymbol{K}_0 \frac{\partial \boldsymbol{r}_0}{\partial X} = \frac{\boldsymbol{R}(X_0 + \delta X) - \boldsymbol{R}(X_0 - \delta X)}{2\delta X} - \frac{\boldsymbol{K}(X_0 + \delta X) - \boldsymbol{K}(X_0 - \delta X)}{2\delta X}\boldsymbol{r}_0 \tag{8}$$

Semi-analytical methods combine ease of implementation and computational efficiency, and they have been implemented in several finite element programs. However, the errors associated with the finite difference approximations of the right-hand-side vector can be substantial.[2,3] Various methods to improve the accuracy of the results have been proposed by several authors.[8–12] In this study the results obtained by the presented procedure will be compared with those obtained by the refined semi-analytical method.[12] Using the latter method, the contribution to the design sensitivities corresponding to the rigid-body motion are evaluated by exact differentiation of the rigid-body modes. Moreover, the non-self-equilibrating components of the element contributions to the pseudoload vector are also corrected using exact differentiation of rigid-body modes. This approach requires only minor programming effort, and the additional computing time is very small.

C. Efficient Finite Difference Derivatives

The evaluation of derivatives by global finite difference approximations involves multiple repeated analyses. In cases where the derivatives must be calculated for many design points or perturbations, the resulting computational effort might be prohibitive. To overcome this problem, it is possible to use fast reanalysis methods to evaluate the displacements $\boldsymbol{r}$. Given an initial design, we assume that the corresponding stiffness matrix $\boldsymbol{K}_0$ is given in the decomposed form of Eq. (4), and the displacements $\boldsymbol{r}_0$ are computed by the initial equilibrium equations [Eq. (3)]. Assume a change in the design and corresponding changes $\Delta\boldsymbol{K}$ and $\Delta\boldsymbol{R}$ in the stiffness matrix and in the load vector, respectively. The modified stiffness matrix $\boldsymbol{K}$ and load vector $\boldsymbol{R}$ are given by

$$\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K}, \qquad \boldsymbol{R} = \boldsymbol{R}_0 + \Delta\boldsymbol{R} \tag{9}$$

Calculation of the modified displacements $\boldsymbol{r}$ by the combined-approximations (CA) method involves the following steps[4]:

1) Calculate the modified values $\boldsymbol{K}$ and $\boldsymbol{R}$ by Eqs. (9).

2) Calculate the basis vectors by the terms of the binomial series

$$\boldsymbol{r}_1 = \boldsymbol{K}_0^{-1}\boldsymbol{R} \tag{10}$$

$$\boldsymbol{r}_i = -\boldsymbol{B}\boldsymbol{r}_{i-1}, \qquad i = 2, 3, \ldots, s \tag{11}$$

where the number of basis vectors s is much smaller than the number of degrees of freedom n, and matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = \boldsymbol{K}_0^{-1}\Delta\boldsymbol{K} \tag{12}$$

Because $\boldsymbol{K}_0$ is given in a decomposed form [Eq. (4)] from initial analysis [Eq. (3)], calculation of the basis vectors involves only forward and backward substitutions. Note that for those cases where $\Delta\boldsymbol{R} = 0$, Eq. (10) can be replaced by $\boldsymbol{r}_1 = \boldsymbol{r}_0$.

3) Calculate the reduced stiffness matrix $\boldsymbol{K}_R$ and load vector $\boldsymbol{R}_R$ by

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B, \qquad \boldsymbol{R}_R = \boldsymbol{r}_B^T\boldsymbol{R} \tag{13}$$

where $\boldsymbol{r}_B$ is the $n \times s$ matrix of the basis vectors $\boldsymbol{r}_B = [\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s]$.

4) Calculate the unknowns $\boldsymbol{y}^T = \{y_1, y_2, \ldots, y_s\}$ by solving the reduced $s \times s$ system

$$\boldsymbol{K}_R\boldsymbol{y} = \boldsymbol{R}_R \tag{14}$$

5) Evaluate the final displacements by the linear combination

$$\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_s\boldsymbol{r}_s = \boldsymbol{r}_B\boldsymbol{y} \tag{15}$$

Efficiency considerations are discussed in Sec. V, accuracy considerations are discussed in the next subsection, and the accuracy of the results for some typical examples is demonstrated in Sec. VI.

D. Accuracy Considerations

It has been shown[13] that exact solutions are obtained efficiently by the CA method for a small number of simultaneous rank-one changes in the stiffness matrix. In such cases, the exact solutions obtained by the CA method and the Sherman–Morison–Woodbury formulas are equivalent.[14] Consider for example the typical case of a rank-one change in the stiffness matrix (e.g., a change in the cross-sectional area of a truss member). An exact solution is obtained by the CA method with only two basis vectors by

$$\boldsymbol{r} = \boldsymbol{r}_1 + y\boldsymbol{r}_2 \tag{16}$$

where $\boldsymbol{r}_1 = \boldsymbol{r}_0$ and $\boldsymbol{r}_2 = -\boldsymbol{B}\boldsymbol{r}_0$ [Eqs. (10) and (11)]. The unknown coefficient y can be determined explicitly by the reduced set of Eq. (14). The two equivalent equations give

$$y = \frac{\boldsymbol{r}_1^T(\boldsymbol{R}_0 - \boldsymbol{K}\boldsymbol{r}_1)}{\boldsymbol{r}_1^T\boldsymbol{K}\boldsymbol{r}_2} \tag{17}$$

In the case of changes in the cross-sectional areas of m truss members, an exact solution is obtained by the CA method with not more than $(m + 1)$ basis vectors.

The CA method provides approximate solutions for high-rank changes in the stiffness matrix, but accurate solutions are often achieved with only a small number of basis vectors. It has been shown[15] that an exact solution is obtained by the CA method in cases where a newly created basis vector becomes a linear combination of the preceding vectors. Thus, it is expected that accurate (nearly exact) solutions will be achieved when the high-order basis vectors come close to being linearly dependent on all previous vectors. Two basis vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}_{i+1}$ are close to being linearly dependent if

$$\cos\beta_{i,i+1} = \left(\boldsymbol{r}_i^T\boldsymbol{r}_{i+1}\right) \Big/ \left(|\boldsymbol{r}_i||\boldsymbol{r}_{i+1}|\right) \approx 1 \tag{18}$$

where $\beta_{i,i+1}$ is the angle between the two vectors. Various numerical examples show that the basis vectors determined by the CA method satisfy the condition of Eq. (18), as the basis vectors index i is increased, even for very large changes in the design.

To evaluate the errors involved in the approximations, we define the error vector of the linear equilibrium equations $\varepsilon(\boldsymbol{R})$

$$\varepsilon(\boldsymbol{R}) = \boldsymbol{K}\boldsymbol{r}(\text{appr}) - \boldsymbol{R} \tag{19}$$

where $\boldsymbol{r}$(appr) is the vector of approximate displacements. Here $\varepsilon(\boldsymbol{R})$ can be looked upon as an imbalanced vector. The norm of $\varepsilon(\boldsymbol{R})$, $\|\varepsilon(\boldsymbol{R})\|$, can be used as a measure of smallness:

$$\|\varepsilon(\boldsymbol{R})\| = [\varepsilon(\boldsymbol{R})^T\varepsilon(\boldsymbol{R})]^{0.5} \tag{20}$$

Although $\varepsilon(\boldsymbol{R})$ might be very small, the error in the solution might still be large. On the other hand, for an accurate solution $\varepsilon(\boldsymbol{R})$ must be small. Therefore, a small residual $\varepsilon(\boldsymbol{R})$ is a necessary but not a sufficient condition for an accurate solution. To obtain more information on the solution errors, the corresponding residual displacements vector is expressed as $\varepsilon(\boldsymbol{r}) = \boldsymbol{K}^{-1}\varepsilon(\boldsymbol{R})$. An analysis can be performed that uses the condition number of $\boldsymbol{K}$, $\psi(\boldsymbol{K})$, to evaluate the solution errors. It has been shown[16] that a large $\psi(\boldsymbol{K})$ means that solution errors are more likely.

The true percentage error E(appr) of the approximate derivatives $\partial\boldsymbol{r}_0/\partial X$(appr) relative to the exact derivatives $\partial\boldsymbol{r}_0/\partial X$(exact) is defined as

$$E(\text{appr}) = 100\frac{\|\partial\boldsymbol{r}_0/\partial X(\text{appr}) - \partial\boldsymbol{r}_0/\partial X(\text{exact})\|}{\|\partial\boldsymbol{r}_0/\partial X(\text{exact})\|} \tag{21}$$

To evaluate the efficiency of the calculations by the method presented, assume that the exact displacements $\boldsymbol{r}_0$ are given for an initial

design X_0. We can distinguish between the following two cases of calculating the sensitivity coefficients:

1) The first case is calculating the sensitivity coefficients at the initial design point represented by X_0, where $\boldsymbol{U}_0$ and $\boldsymbol{r}_0$ are known from the initial analysis. Sensitivity analysis by the forward-difference method involves an additional complete analysis (for $X_0 + \delta X$), whereas the central difference method requires two additional complete analyses (for $X_0 - \delta X$ and $X_0 + \delta X$). Using the CA method, on the other hand, the main computational effort is involved in calculation of the basis vectors by forward and backward substitutions for the points where analysis is required. Because the points $X_0 - \delta X$ and $X_0 + \delta X$ represent small changes in the initial design X_0, high accuracy of the approximations is expected with a small number of basis vectors.

2) The second case is calculating the sensitivity coefficients at a design point represented by some X, where the factorized stiffness matrix $\boldsymbol{U}$ and the exact displacements $\boldsymbol{r}$ are not known. In this case, both the forward-difference and the central-difference methods involve two additional complete analysis (for X and $X + \delta X$, and for $X - \delta X$ and $X + \delta X$, respectively). Using the CA method, the basis vectors are calculated by forward and backward substitutions for the points where analysis is required. In cases where the points $X - \delta X$ and $X + \delta X$ represent large changes in the initial design X_0, the accuracy of the approximations might not be as good as for small changes. However, as noted earlier, for a rank-one change in the stiffness matrix, exact solutions are obtained [Eqs. (16) and (17)].

IV. Sensitivity Calculations for Eigenproblems

A. Analytical Derivatives

Consider the initial eigenproblem

$$\boldsymbol{K}_0\boldsymbol{r}_0 = \lambda_0\boldsymbol{M}_0\boldsymbol{r}_0 \tag{22}$$

where $\boldsymbol{M}_0$ is the mass matrix, λ_0 is the eigenvalue representing the free vibration frequency squared, and $\boldsymbol{r}_0$ is the eigenvector representing the corresponding mode shape. The latter is often normalized such that

$$\boldsymbol{r}_0^T\boldsymbol{M}_0\boldsymbol{r}_0 = 1 \tag{23}$$

Differentiating Eqs. (22) and (23) with respect to a design variable X and rearranging gives

$$(\boldsymbol{K}_0 - \lambda_0\boldsymbol{M}_0)\frac{\partial \boldsymbol{r}_0}{\partial X} - \frac{\partial \lambda_0}{\partial X}\boldsymbol{M}_0\boldsymbol{r}_0 = -\left(\frac{\partial \boldsymbol{K}_0}{\partial X} - \lambda_0\frac{\partial \boldsymbol{M}_0}{\partial X}\right)\boldsymbol{r}_0$$

$$\boldsymbol{r}_0^T\boldsymbol{M}_0\frac{\partial \boldsymbol{r}_0}{\partial X} = -\frac{1}{2}\boldsymbol{r}_0^T\frac{\partial \boldsymbol{M}_0}{\partial X}\boldsymbol{r}_0 \tag{24}$$

or, in matrix form,

$$\begin{bmatrix} \boldsymbol{K}_0 - \lambda_0\boldsymbol{M}_0 & -\boldsymbol{M}_0\boldsymbol{r}_0 \\ \boldsymbol{r}_0^T\boldsymbol{M}_0 & 0 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \boldsymbol{r}_0}{\partial X} \\ \dfrac{\partial \lambda_0}{\partial X} \end{Bmatrix} = -\begin{Bmatrix} \left(\dfrac{\partial \boldsymbol{K}_0}{\partial X} - \lambda_0\dfrac{\partial \boldsymbol{M}_0}{\partial X}\right)\boldsymbol{r}_0 \\ \dfrac{1}{2}\boldsymbol{r}_0^T\dfrac{\partial \boldsymbol{M}_0}{\partial X}\boldsymbol{r}_0 \end{Bmatrix} \tag{25}$$

In the solution of Eq. (25), care must be taken because the principal minor $(\boldsymbol{K}_0 - \lambda_0\boldsymbol{M}_0)$ is singular. In many cases we are interested only in the derivatives of the eigenvalues. These derivatives can be obtained by premultiplying the first equation (24) by $\boldsymbol{r}_0^T$ and rearranging to obtain

$$\frac{\partial \lambda_0}{\partial X} = \boldsymbol{r}_0^T\left(\frac{\partial \boldsymbol{K}_0}{\partial X} - \lambda_0\frac{\partial \boldsymbol{M}_0}{\partial X}\right)\boldsymbol{r}_0 \Big/ \boldsymbol{r}_0^T\boldsymbol{M}_0\boldsymbol{r}_0 \tag{26}$$

Note that this is only correct if the eigenvalue λ_0 is distinct. Several methods have been proposed to calculate the derivatives of the eigenvectors.[1] An efficient solution procedure using finite difference and approximate reanalysis by the CA method is described in the next section.

B. Efficient Finite Difference Derivatives

Eigenproblem reanalysis by the CA method has been discussed in detail in previous studies.[17,18] In this section the solution procedure is briefly described. Given an initial design, we assume that the corresponding stiffness matrix $\boldsymbol{K}_0$ is given in the decomposed form of Eq. (4), and the corresponding eigenvectors $\boldsymbol{r}_0$ and eigenvalues λ_0 are computed by solving the initial eigenproblem [Eq. (22)]. Assume a change in the design and corresponding changes $\Delta\boldsymbol{K}$ in the stiffness matrix and $\Delta\boldsymbol{M}$ in the mass matrix, respectively. The modified matrices are given by

$$\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K}, \qquad \boldsymbol{M} = \boldsymbol{M}_0 + \Delta\boldsymbol{M} \tag{27}$$

The object is to estimate efficiently and accurately the modified eigenvectors $\boldsymbol{r}$ and corresponding eigenvalues λ without solving the complete set of modified equations

$$(\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{r} = \lambda\boldsymbol{M}\boldsymbol{r} \tag{28}$$

The solution process involves the following steps:

1) Calculate the modified values $\boldsymbol{K}$ and $\boldsymbol{M}$ [Eqs. (27)].

2) Calculate the basis vectors by the terms of the binomial series [see Eqs. (10) and (11)]

$$\boldsymbol{r}_1 = \boldsymbol{K}_0^{-1}\boldsymbol{M}\boldsymbol{r}_0, \qquad \boldsymbol{r}_i = -\boldsymbol{B}\boldsymbol{r}_{i-1}, \qquad i = 2, 3, \ldots, s \tag{29}$$

where $\boldsymbol{B}$ is defined by Eq. (12). For any mode shape we use the corresponding initial mode $\boldsymbol{r}_0$ for the first basis vector. Calculation of the basis vectors involves only forward and backward substitutions. Assume that we have calculated the first m eigenvectors $\boldsymbol{r}(1), \boldsymbol{r}(2), \ldots, \boldsymbol{r}(m)$ and that we want to obtain the eigenvector $\boldsymbol{r}(m+1)$, which is $\boldsymbol{M}$ orthogonal to the eigenvectors $\boldsymbol{r}(i)(i = 1, \ldots, m)$. Using the Gram–Schmidt orthogonalization, we calculate the coefficients

$$\alpha_i = \boldsymbol{r}(i)^T\boldsymbol{M}\bar{\boldsymbol{r}}(m+1) \tag{30}$$

where $\bar{\boldsymbol{r}}(m+1)$ is the nonorthogonal vector. The orthogonal vector $\boldsymbol{r}(m+1)$ is obtained by

$$\boldsymbol{r}(m+1) = \bar{\boldsymbol{r}}(m+1) - \sum_{i=1}^{m}\alpha_i\boldsymbol{r}(i) \tag{31}$$

It was found that using the Gram–Schmidt orthogonalization of Eqs. (30) and (31) for all basis vectors improves significantly the accuracy of the results.

3) Calculate the reduced matrices $\boldsymbol{K}_R$ and $\boldsymbol{M}_R$ by

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B, \qquad \boldsymbol{M}_R = \boldsymbol{r}_B^T\boldsymbol{M}\boldsymbol{r}_B \tag{32}$$

4) Solve the reduced $s \times s$ eigenproblem for the first eigenvalue λ and mode shape $\boldsymbol{y}$

$$\boldsymbol{K}_R\dot{\boldsymbol{y}} = \lambda\boldsymbol{M}_R\boldsymbol{y} \tag{33}$$

5) Evaluate the eigenvector by the linear combination

$$\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_s\boldsymbol{r}_s = \boldsymbol{r}_B\boldsymbol{y} \tag{34}$$

Calculation of the sensitivity coefficients for several mode shapes by the procedure presented is demonstrated in example 3, Sec. VI.

V. Efficiency Considerations

The efficiency of reanalysis by the CA method, compared with complete analysis of the modified design, can be measured by various criteria, for example, the CPU effort or the number of algebraic operations. It is then possible to relate the computational effort to various parameters such as the number of degrees of freedom, the bandwidth of the stiffness matrix, the number of basis vectors considered, and the accuracy of the results. In general, larger savings will be achieved for large-scale structures having many degrees of freedom. More significant savings are expected for problems of nonlinear and dynamic analysis.

It was found that the number of algebraic operations and the total CPU effort involved in the CA method are usually much smaller than those needed to carry out complete analysis of the modified design. In the CA method, the main computational effort is invested in calculation of the basis vectors. As noted earlier, calculation of each additional vector involves only forward and backward substitutions. In many problems a small number of basis vectors are sufficient to achieve accurate results. Solution of various problems indicated that calculation of each basis vector involves about 1–2% of the CPU time needed for complete analysis. In some common cases, repeated calculation of the basis vectors involves almost no computational effort. In the typical case of a rank-one change in the stiffness matrix, an exact solution is obtained by calculation of only a single basis vector [Eqs. (16) and (17)]. It was found that in this case the total CPU effort is reduced by more than 95%, compared with complete analysis of the modified design. For general midsize problems the total CPU effort can be reduced by more than 75%, compared with complete analysis of the modified design. More significant reductions could be achieved for large-scale problems.

VI. Numerical Examples

In all examples the forward-difference method has been used, and arbitrary units have been assumed. Small-scale examples are presented, but similar results have been obtained for larger problems.

A. Example 1: Cantilever Beam

The beam example shown in Fig. 1 has been studied by several authors. The beam consisting of n equally sized beam elements is clamped on one side and loaded by a unit moment at the other side. The uniform bending stiffness is EI and length X of the elements are taken as the design variable. It has been shown[3] that the sensitivity errors according to the traditional semi-analytical method are proportional to ηn^2, where $\eta = \delta X / X_0$ denotes the relative perturbation of the design variable, δX is the perturbation, and X_0 is the initial length. Results are calculated by the following methods: $\partial \boldsymbol{r}/\partial X$ (exact) = exact analytical derivatives, $\partial \boldsymbol{r}/\partial X$ (FD) = finite difference (FD) derivatives using exact analysis, $\partial \boldsymbol{r}/\partial X$ (CAs) = finite difference derivatives using CA with s basis vectors, and $\partial \boldsymbol{r}/\partial X$ (RSA) = refined semi-analytical (RSA) derivatives, as presented in Ref. 12.

Assume the initial design represented by $EI_0 = X_0 = 1.0$. Derivatives and the true percentage errors [Eq. (21)] for $n = 6$, $\delta X = 0.01$ by the various methods are shown in Table 1. It is observed that

Table 1 Derivatives $\partial r/\partial X$ for $n = 6$, $\delta X = 0.01$, cantilever beam

	Method			
Variable	Exact	FD	CA3	RSA
$\partial \boldsymbol{r}/\partial X$	1.00	1.01	1.00	0.99
	1.00	1.00	1.00	0.99
	4.00	4.02	4.01	3.98
	2.00	2.00	2.00	1.98
	9.00	9.05	9.03	8.95
	3.00	3.00	3.00	2.97
	16.00	16.08	16.05	15.92
	4.00	4.00	4.00	3.96
	25.00	25.13	25.09	24.88
	5.00	5.00	5.00	4.95
	36.00	36.18	36.15	35.82
	6.00	6.00	6.00	5.94
E (appr)	——	0.49	0.39	0.52

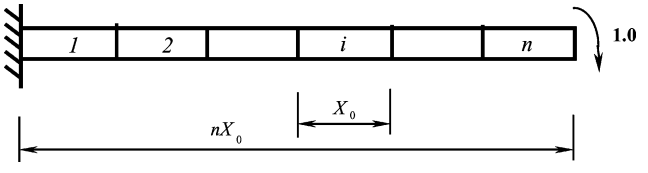

Fig. 1 Cantilever beam.

Table 2 True percentage errors [Eq. (21)] for various cases, cantilever beam

n	s	δX	E(FD)	E(CAs)	E(RSA)
4	3	0.01	0.48	0.45	0.55
		0.001	0.048	0.048	0.055
		0.0001	0.0048	0.0048	0.0055
6	3	0.01	0.49	0.39	0.52
		0.001	0.049	0.048	0.053
		0.0001	0.0049	0.0049	0.0053
10	4	0.01	0.50	0.44	0.51
		0.001	0.050	0.050	0.051
		0.0001	0.0050	0.0050	0.0051
20	5	0.01	0.50	0.49	0.50
		0.001	0.050	0.050	0.050
		0.0001	0.0050	0.0050	0.0050
40	5	0.01	0.50	0.45	0.50
		0.001	0.050	0.050	0.050
		0.0001	0.0050	0.0050	0.0050

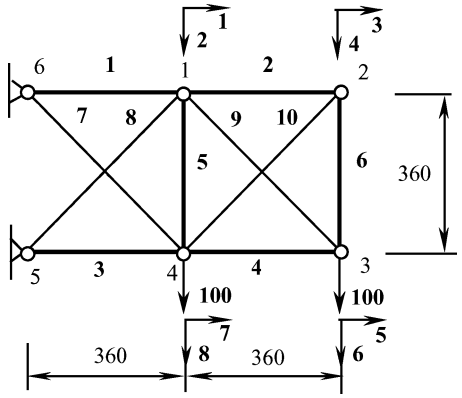

Fig. 2 Ten-bar truss.

the results achieved by finite difference derivatives using the CA method are very close to those obtained by exact analysis.

Assume the following 15 cases of n and δX:

$$n = 4, 6, 10, 20, 40, \qquad \delta X = 0.01, 0.001, 0.0001$$

The true percentage errors [Eq. (21)] for all cases are summarized in Table 2. It is observed that better accuracy is obtained for smaller perturbations δX; the errors obtained by the FD, the CA, and the RSA methods are similar; and for larger numbers of elements n, larger numbers of basis vectors s are required to obtain small errors by the CA method.

B. Example 2: Truss

Consider the 10-bar truss shown in Fig. 2. The modulus of elasticity is 30,000 and the eight analysis unknowns are the horizontal and the vertical displacements at joints 1, 2, 3, and 4, respectively. The design variables are the cross-sectional areas $\boldsymbol{X}$, and the initial cross sections $\boldsymbol{X}_0$ are all unity. Assuming that results of exact displacements are given only at the initial point $\boldsymbol{X}_0$, the following cases have been solved:

1) The first case is calculation of sensitivities $\partial \boldsymbol{r}/\partial X_1$ and $\partial \boldsymbol{r}/\partial X_2$ at $\boldsymbol{X}_0$ based on analysis at $\delta X_1 = 0.001$ and $\delta X_2 = 0.001$, respectively. For such changes in a single member, an exact solution is obtained by Eqs. (16) and (17). Exact analytical derivatives $\partial \boldsymbol{r}/\partial X$(exact), finite difference derivatives using exact analysis $\partial \boldsymbol{r}/\partial X$(FD), and finite difference derivatives using approximate analysis with a single basis vector $\partial \boldsymbol{r}/\partial X$(CA1) are shown in Tables 3 and 4. It is observed that the results of $\partial \boldsymbol{r}/\partial X$(FD) and $\partial \boldsymbol{r}/\partial X$(CA1) are identical, with very small errors compared with $\partial \boldsymbol{r}/\partial X$(exact).

2) For the second case, assume the line from the initial design given by

$$\boldsymbol{X} = \boldsymbol{X}_0 + \alpha \Delta \boldsymbol{X}_0$$

Table 3 Sensitivities $\partial r/\partial X_1$ at $X = 1.0$, $\delta X_1 = 0.001$, 10-bar truss

Method	
Exact	FD = CA1
−2.0703	−2.0685
−1.2950	−1.2939
−2.0987	−2.0968
−3.5024	−3.4993
0.2457	0.2455
−3.5308	−3.5276
0.2741	0.2738
−1.0493	−1.0484

Table 4 Sensitivities $\partial r/\partial X_2$ at $X = 1.0$, $\delta X_2 = 0.001$, 10-bar truss

Method	
Exact	FD = CA1
−0.0058	−0.0058
−0.0223	−0.0223
−0.4369	−0.4365
−0.2660	−0.2657
0.0446	0.0446
−0.2155	−0.2153
−0.0058	−0.0058
0.0223	0.0223

Table 5 Sensitivities $\partial r/\partial \alpha$ at $\alpha = 0$, $\delta\alpha = 0.0001$, 10-bar truss

Method		
Exact	FD	CA2
−16.548	−16.536	−16.536
−32.028	−32.009	−32.008
−17.541	−17.53	−17.53
−70.619	−70.575	−70.576
17.733	17.721	17.722
−71.612	−71.568	−71.569
17.052	17.040	17.041
−33.214	−33.194	−33.194

Table 6 Sensitivities $\partial r/\partial \alpha$ at $\alpha = 0.5$, $\delta\alpha = 0.0001$, 10-bar truss

Method		
Exact	FD	CA2
−0.79	−0.79	−0.80
−2.09	−2.09	−2.16
−1.01	−1.01	−1.05
−4.89	−4.89	−4.96
1.24	1.24	1.24
−5.10	−5.10	−5.20
0.86	0.86	0.85
−2.42	−2.42	−2.37

where α is a variable representing the step size and $\Delta \boldsymbol{X}_0$ is defined as

$$\Delta \boldsymbol{X}_0^T = \{7.0, -0.999, 7.0, 3.0, -0.999,$$
$$-0.999, 4.667, 4.667, 4.667, -0.999\}$$

Exact analytical derivatives $\partial \boldsymbol{r}/\partial X$(exact) at $\alpha = 0$, finite difference derivatives $\partial \boldsymbol{r}/\partial X$(FD) using exact analysis at $\alpha = 0$ and at $\alpha = 0.0001$, and finite difference derivatives $\partial \boldsymbol{r}/\partial X$(CA2) using approximate analysis at $\alpha = 0.0001$ and only two basis vectors are shown in Table 5. It is observed that the results of $\partial \boldsymbol{r}/\partial \alpha$(FD) and $\partial \boldsymbol{r}/\partial \alpha$(CA2) are practically identical, with small errors compared with $\partial \boldsymbol{r}/\partial \alpha$(exact).

3) The following results are shown in Table 6: exact analytical derivatives $\partial \boldsymbol{r}/\partial X$(exact) at $\alpha = 0.5$; finite difference derivatives $\partial \boldsymbol{r}/\partial X$(FD) using exact analysis at $\alpha = 0.5$ and at $\alpha = 0.5001$; finite difference derivatives $\partial \boldsymbol{r}/\partial \alpha$(CA2) using approximate analysis at $\alpha = 0.5$ and at $\alpha = 0.5001$ and only two basis vectors. It is observed that the errors for this very large change in the design are still reasonable.

C. Example 3: Frame

Consider the eight-story frame shown in Fig. 3. The mass of the frame is lumped in the girders, with initial values $M_1 = 1.0$, $M_2 = 1.5$, and $M_3 = 2.0$. The girders are assumed to be nondeformable, and the initial lateral stiffness of each of the stories is given by $EI/L^3 = 5.0$. Assume a single design variable, the lateral stiffness of the bottom story, $X = EI/L^3$, and a perturbation of $\delta X = 0.01$. Exact analytical derivatives and forward-difference derivatives based on exact analysis and on approximate analysis with only two basis vectors for the first three mode shapes are shown in Tables 7–9 and in Fig. 4. It is observed that very small errors are obtained for the eigenvalue derivatives $\partial\lambda/\partial X$. Larger errors are obtained for the eigenvector derivatives $\partial \boldsymbol{r}/\partial X$.

Table 7 Derivatives of first mode shape, $\delta X = 0.01$, eight-story frame

	Method		
Variable	Exact	FD	CA2
$100\partial \boldsymbol{r}/\partial X$	0.61	0.61	0.55
	0.54	0.54	0.49
	0.39	0.39	0.37
	0.16	0.16	0.16
	−0.16	−0.16	−0.12
	−0.53	−0.53	−0.47
	−0.92	−0.92	−0.87
	−1.32	−1.32	−1.31
$\partial\lambda/\partial X$	0.0831	0.0830	0.0830

Table 8 Derivatives of second mode shape, $\delta X = 0.01$, eight-story frame

	Method		
Variable	Exact	FD	CA2
$100\partial \boldsymbol{r}/\partial X$	0.58	0.58	0.20
	0.00	0.00	−0.19
	−0.96	−0.96	−0.85
	−1.92	−1.91	−1.57
	−2.22	−2.22	−1.92
	−1.45	−1.45	−1.47
	0.37	0.37	0.00
	2.88	2.88	2.81
$\partial\lambda/\partial X$	0.6711	0.6703	0.6703

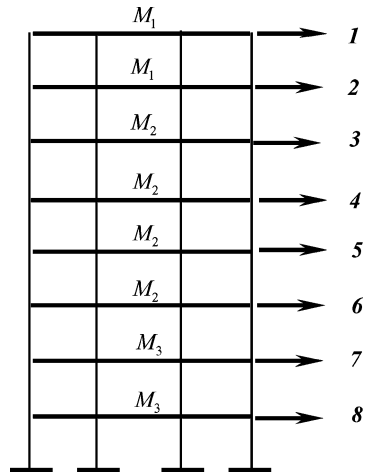

Fig. 3 Eight-story frame.

Table 9 Derivatives of third mode shape, $\delta X = 0.01$, eight-story frame

	Method		
Variable	Exact	FD	CA2
$100\partial \boldsymbol{r}/\partial X$	−0.06	−0.06	−0.46
	−0.87	−0.87	−0.89
	−1.64	−1.64	−1.24
	−0.76	−0.76	−0.52
	1.77	1.77	1.45
	3.83	3.83	3.43
	2.98	2.98	3.25
	−2.09	−2.09	−1.99
$\partial\lambda/\partial X$	1.1346	1.1339	1.1338

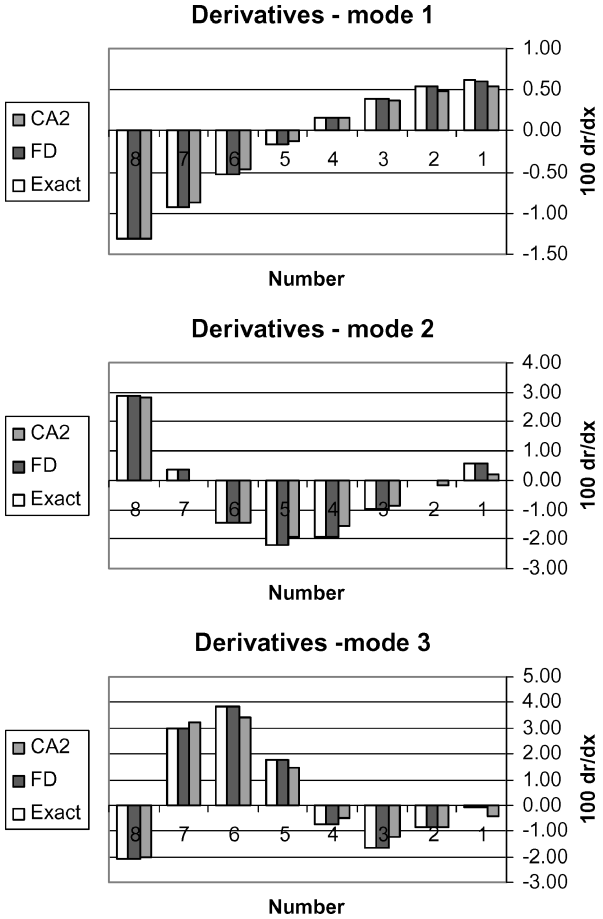

Fig. 4 Derivatives of first three mode shapes, $\delta X = 0.01$, eight-story frame.

VII. Conclusions

In this study an efficient solution procedure for evaluating finite difference design sensitivities is presented. Instead of repeating the complete analysis calculations for each perturbed design, the displacements corresponding to various modified designs are evaluated by the recently developed combined approximations method. Calculation of the derivatives can be performed also for designs where the exact displacement response is not known. Given the displacements for an initial design, calculations of displacements for various perturbed design points involve mainly forward and backward substitutions. As a result, the efficiency of the calculations is significantly improved, particularly when a large number of sensitivity coefficients must be determined.

The solution procedure is demonstrated for sensitivity calculations in static problems as well as eigenproblems. It is shown that the accuracy of the results obtained by the presented procedure is similar to the accuracy obtained by finite difference calculations based on exact analysis. In all examples the forward-difference method has been used. Better accuracy could be achieved by using the central-difference approximation and considering additional terms of the combined approximations. Good accuracy is demonstrated also for some problems where semi-analytical methods might provide poor results. The accuracy in such problems is similar to that obtained by a refined semi-analytical method. The solution procedure is easy to implement. Yet, it might not be suitable in cases where finite difference approximations involve accuracy problems.

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